Variation iteration method (VIM) for solving some models of nonlinear diffusion problems

Kholoud A.Abufalgha*1,Fatma A.Zaggout2
[1]Mathematics Department, Faculty of Science, Misurata University
[2] Mathematics Department, Faculty of Science, Misurata University
kholoud.abufalgha@gmail.com

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Abstract:
In this work we will apply He's Variation iteration method (VIM) to investigate three physical models of nonlinear partial differential equations. These models describe the nonlinear diffusion equations in one dimension. Using this technique we obtain an exact on an approximate solution. The efficiency and the ability of the approach will be shown by applying the procedure on some examples. The computations associated with the examples in this paper were performed using Matlab 17.

Keywords:*Variation iteration method-nonlinear diffusion equations-partial differential equations.*

1-Introduction:

Diffusion problems linear or nonlinear appear in scientific applications in engineering, physics and dynamicl models. Several methods are applied to find the solutions of these problems. One of these methods is the Variation iteration method (VIM). This method was first introduced by He in 1999 [1]. In this method the solution is considered using lagrange multiplier and starting with zeroth approximation as which successive approximations usually converge rapidly to an accurate approximation of the exact solution. Considerable research works have been conducted recently in applying this method to a class of linear and nonlinear equations [2,4,5,6].

The nonlinear Diffusion equation is a partial differential equation of second order that can be written as

$$\partial_t u = \sum_{i=1}^{n} \partial_{x_i}\big(A_i(x,t,u,\nabla u)\big) + B(x,t,u,\nabla u)$$

Where $u = u(x,t)$, $\nabla u = \left(\partial_{x_i}u, \partial_{x_2}u, \dots, \partial_{x_n}u\right)$,$x = (x_1, x_2, \dots, x_n)$ and A,B are functions in x,t,u,$\nabla u$and nonlinear in u or ∇u[7].

2-Examples of nonlinear diffusion equations:

In this paper, three models of nonlinear diffusion equations have been studied:

1- **Burger's equation:**

$$u_t + uu_x = Du_{xx} \quad ; \quad x \in R\,, t > 0$$

Where D is the diffusion coefficient; $D = \frac{1}{Re}$, Re is the Reynoldsnumber[9] which describes the diffusion in one dimension[8].

2- **Fisher's equation:**

$$u_t - vu_{xx} = au(1-u) \;\; ; \quad x \in R\,, t > 0$$

a,v are constants[8].

3- **Porous Medium equation [PME][8]:**

$$u_t = a\frac{\partial}{\partial x}\left(u^n\frac{\partial u}{\partial x}\right) \quad ; \quad x \in R\,, t > 0$$

3-Description of the Variation iteration method:

In order to introduce a simple idea on the VIM, we consider the differential equation

$$Lu(x,t) + Nu(x,t) = g(x,t) \qquad (1)$$

Where L is a linear operator, N a nonlinear operator and $g(x,t)$ an inhomogeneous term.the variational iteration method was proposed by He, where a correction functional for Eq. (1) can be written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\{Lu_n(x,\tau) + N\tilde{u}_n(x,\tau) - g(x,\tau)\}\,d\tau$$

It is obvious that the successive approximations $u_j, j \geq 0$, can be established by determiningλ, a general Lagrange multiplier, which can be identified optimally via the variational theory. The function $\tilde{u}_n$ is a restricted variation, which means $\delta\tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplierλ, that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t)\,, n \geq 0$, of the solution $u(x,t)$will be readily obtained upon using the Lagrange multiplier obtained and by using any selective function u_0. The initial values $u(x,0)$and $u_t(x,0)$ are usually used for selecting the zeroth approximation u_0. With λ detemined, then several approximations $u_j(x,t), j \geq 0$,follow immediately[1,4,5]. Consequently, the exact solution may be obtained by using

$$u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

4-Convergence of Variation iteration method:

Here, we will study the convergence analysis as the same manner of the variational iteration method to the nonlinear equations. Let us consider the Banach space X , with the setof applications, $u{:}\,\Omega \to \mathcal{R}$, with

$$\int_\Omega u^2\,dx < \infty$$

and the associated norm[4] : $\|u\|^2 = \int_\Omega u^2\,dx$

The VIM is convergent if the conditions of the following theorem are satisfied.

Theorem 1: (Banach's fixed-point theorem)

Assume that X be a Banach space and $A{:}\,X \to X$is a nonlinear mapping, and suppose that$\|A[u] - [\bar{u}]\| \leq \gamma\|u - \bar{u}\|$for some constant$\gamma < 1$. Then A has a unique fixed point.

According to the theorem 1, for the nonlinear mapping

$$A[u] = u(x,t) + \int_0^t \lambda\left[F(u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial \tau},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial \tau^2},\frac{\partial^2 u}{\partial x\partial \tau})\right]d\tau$$

A sufficient condition for convergence of the variational iteration method is the strictlycontraction of A, such that for $u, \bar{u} \in X$we have $\|u\| \leq M$and $\|\bar{u}\| \leq M$, for $M > 0$ [4].

5-Numerical examples:

This section contains three examples of nonlinear diffusion equations in one dimension.

Example1:
Consider the following nonlinear Burger's equation with the exact solution [11] $u(x,t)=\frac{2x}{1+2t}$,

$$u_t - u_{xx} + uu_x = 0$$

The initial condition

$$u(x,0) = 2x$$

Solution:
According to the VIM, we can construct the correction functional of equation as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\{(u_n)_\tau - (u_n)_{xx} + u_n(u_n)_x\}\, d\tau$$

To find optimal value of λ, we have

$$\lambda(\tau) = \frac{-(t-\tau)^{s-1}}{(s-1)!} = \frac{-(t-\tau)^{1-1}}{(1-1)!} = -1$$

Consequently, the following approximations are obtained using the above iteration formula started with the initial approximations

$$u_0(x,t) = 2x$$
$$u_1(x,t) = 2x(1-2t)$$
$$u_2(x,t) = 2x\left(1 - 2t + (2t)^2 - \frac{8}{3}t^3\right)$$
$$u_3(x,t) = 2x(1 - 2t + (2t)^2 - (2t)^3 \cdots)$$
$$u_4(x,t) = 2x(1 - 2t + (2t)^2 - (2t)^3 + (2t)^4 \cdots)$$
$$u_5(x,t) = 2x(1 - 2t + (2t)^2 - (2t)^3 + (2t)^4 - (2t)^5 \cdots)$$
$$\vdots$$
$$u_n(x,t) = 2x(1 - 2t + (2t)^2 - (2t)^3 + (2t)^4 - (2t)^5 + \cdots)$$

The solution in a closed form is found to be

$$u(x,t) = \lim_{n\to\infty} u_n(x,t) = \frac{2x}{1+2t}$$

Figures (1),(2),(3),(4),(5) and Table(1) show the results .

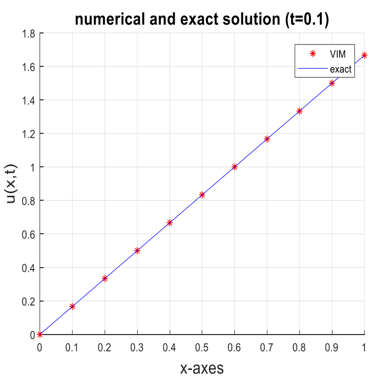


Figure (1)

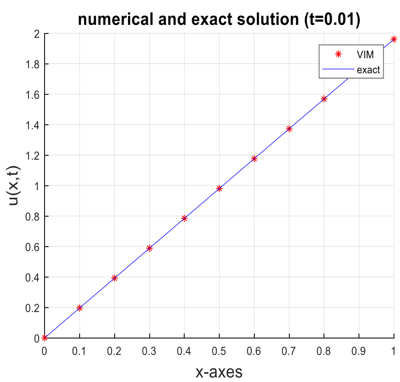


Figure (2)

We will see the effectiveness and accuracy of the VIM method through the following numerical results:

Table (1) shows the absolute error between the exact solution and numerical solution of example (1)

$t = 0.1$ x	u_{VIM}	u_{exact}	*Absolute error*
0	0	0	0
0.1	0.166656	0.166666666	0.000010666
0.3	0.499968	0.5	0.000032
0.5	0.83328	0.833333333	0.000053333
0.7	1.166592	1.166666667	0.000074666
0.9	1.499904	1.5	0.000096
1	1.66656	1.666666667	0.000106666
$t = 0.01$ x	u_{VIM}	u_{exact}	*Absolute error*
0	0	0	0
0.1	0.196078431	0.196078431	1.254×10^{-11}
0.3	0.588235294	0.588235294	3.764×10^{-11}
0.5	0.980392156	0.980392156	6.274×10^{-11}
0.7	1.37254902	1.37254902	8.78×10^{-11}
0.9	1.764705882	1.764705882	1.129×10^{-10}
1	1.960784314	1.960784314	1.254×10^{-10}

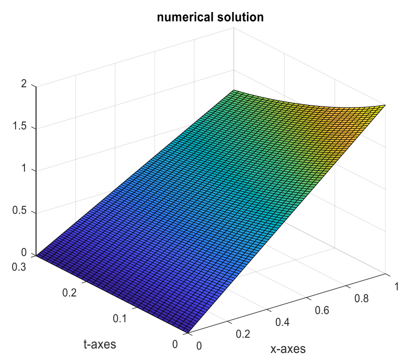


Figure (3) : $\mathbf{0} \leq x \leq \mathbf{1}, \mathbf{0} \leq t < 1$

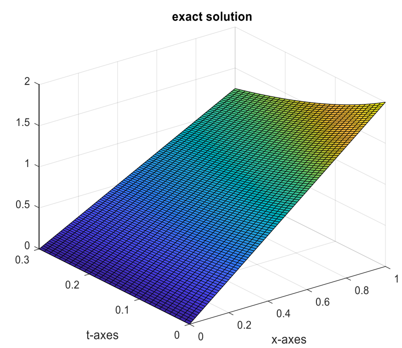


Figure (4): $\mathbf{0} \leq x \leq \mathbf{1}, \mathbf{0} \leq t < 1$

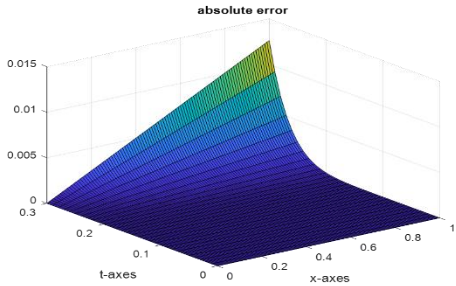


Figure (5) :$0 \leq x \leq 1$, $0 \leq t < 1$

Example2:
Consider the following nonlinear Fisher's equation ,

$$u_t - u_{xx} - u(1-u) = 0$$

The initial condition

$$u(x,0) = \beta$$

Solution:
According to the VIM, we can construct the correction functional of equation as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\{(u_n)_\tau - (u_n)_{xx} - u_n(1-u_n)\}\,d\tau$$

To find optimal value of λ, we have

$$\lambda(\tau) = \frac{-(t-\tau)^{s-1}}{(s-1)!} = \frac{-(t-\tau)^{1-1}}{(1-1)!} = -1$$

Consequently, the following approximations are obtained using the above iteration formula started with the initial approximations

$$u_0(x,t) = \beta$$
$$u_1(x,t) = \beta + \beta(1-\beta)t$$
$$u_2(x,t) = \beta + \beta(1-\beta)t + \frac{t^2}{2!}\beta(1-3\beta+2\beta^2) - \frac{t^3}{3}\beta^2(\beta-1)^2$$
$$u_3(x,t) = \beta + \beta(1-\beta)t + \frac{t^2}{2!}\beta(1-3\beta+2\beta^2) + \frac{t^3}{3!}(\beta-6\beta^2-10\beta^3+5\beta^4) + \frac{t^4}{3}(\beta-1)^2\beta^2(2\beta-1) - \frac{t^5}{60}(\beta-1)^2\beta^2(3-20\beta+20\beta^2) + \frac{t^6}{18}(\beta-1)^3\beta^3(2\beta-1) - \frac{t^7}{63}(\beta-1)^4\beta^4$$

⋮

Doing some algebraic operations leads us to exprees the solution in the following closed form:

$$u(x,t) = \frac{\beta e^t}{1-\beta+\beta e^t}$$

Figures (6),(7),(8),(9) and Table(2) show the results .

Table (2) shows the absolute error between the exact solution and numerical solution of example (2)

$t = 0.1$ $\beta = 0.2$ x	u_{VIM}	u_{exact}	*Absolute error*
0	0.2165	0.2165	0
0.1	0.2165	0.2165	0
0.3	0.2165	0.2165	0
0.5	0.2165	0.2165	0
0.7	0.2165	0.2165	0
0.9	0.2165	0.2165	0
1	0.2165	0.2165	0

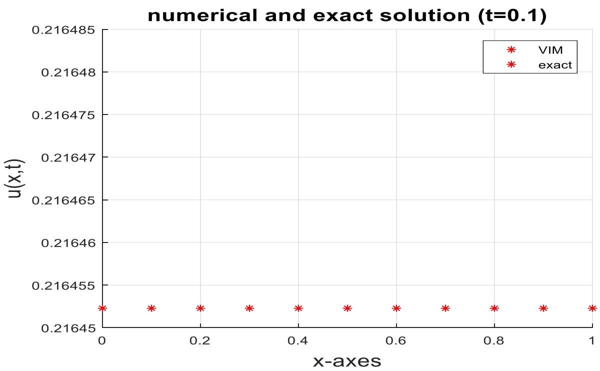
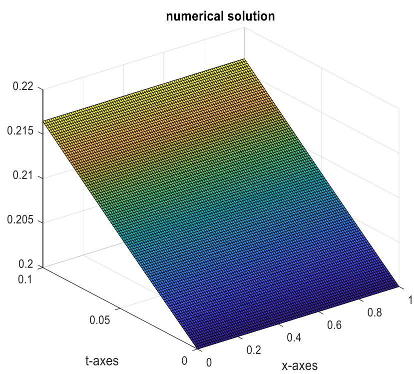


Figure (6)



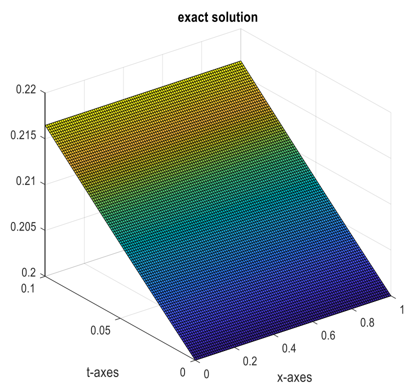


Figure (7): $0 \leq x \leq 1, 0 \leq t \leq 1$ **Figure (8):** $0 \leq x \leq 1, 0 \leq t \leq 1$

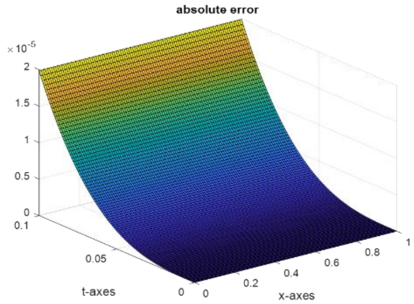


Figure (9): $\mathbf{0 \le x \le 1, 0 \le t \le 1}$

Example3:

Consider the following nonlinear Porous Medium equation [PME] with the exact solution[3],[10], $u(x,t) = x + t$,

$$u_t = \frac{\partial}{\partial x}(uu_x)$$

The initial condition

$$u(x,0) = x \quad ; \quad 0 \le x \le 1$$

And boundary condition

$$u(0,t) = t \quad , \quad u(1,t) = 1 + t \quad ; \quad t > 0$$

Solution:

According to the VIM, we can construct the correction functional of equation as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\{(u_n)_\tau - u_n(u_n)_{xx} - (u_n)_x^2\}\, d\tau$$

To find optimal value of λ, we have

$$\lambda(\tau) = \frac{-(t-\tau)^{s-1}}{(s-1)!} = \frac{-(t-\tau)^{1-1}}{(1-1)!} = -1$$

Consequently, the following approximations are obtained using the above iteration formula started with the initial approximations

$u_0(x,t) = x$

$u_1(x,t) = x + t$

$u_2(x,t) = x + t$

$u_3(x,t) = x + t$

⋮

$u_n(x,t) = x + t$

The solution in a closed form is readily found to be

$u(x,t) = \lim_{n\to\infty} u_n(x,t) = x + t$

Figures (10),(11),(12),(13),(14) and Table(3) show the results .

Table (3) shows the absolute error between the exact solution and numerical solution of example (3)

$t=0.1$ x	u_{VIM}	u_{exact}	*Absolute error*
0	0.1	0.1	0
0.1	0.2	0.2	0
0.3	0.4	0.4	0
0.5	0.6	0.6	0
0.7	0.8	0.8	0
0.9	1	1	0
1	1.1	1.1	0
$t=0.01$ x	u_{VIM}	u_{exact}	*Absolute error*
0	0.01	0.01	0
0.1	0.11	0.11	0
0.3	0.31	0.31	0
0.5	0.51	0.51	0
0.7	0.71	0.71	0
0.9	0.91	0.91	0
1	1.01	1.01	0

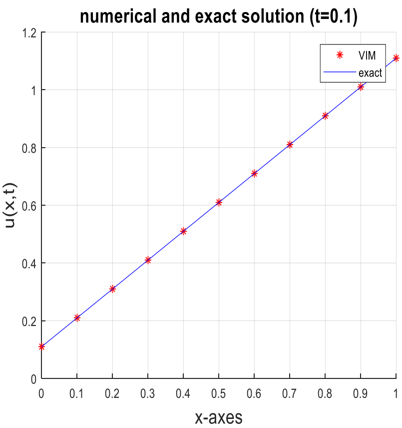


Figure (10)

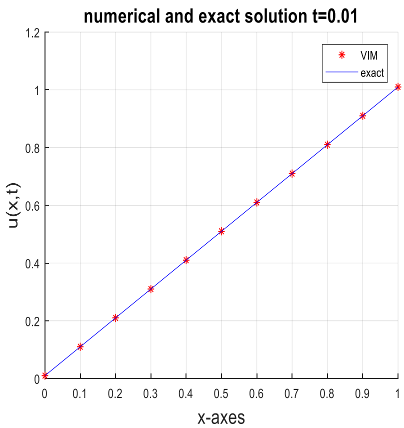
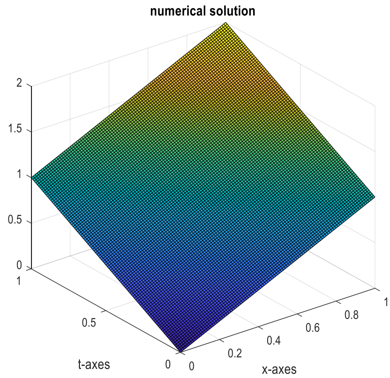


Figure (11)



Figure(12) : $0 \leq x \leq 1, 0 \leq t \leq 1$

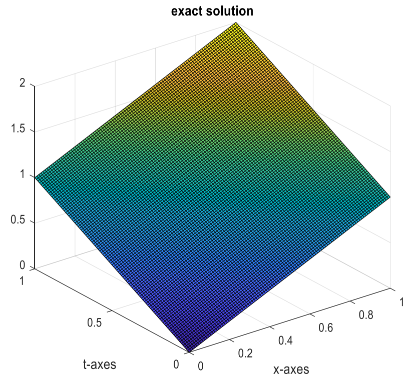
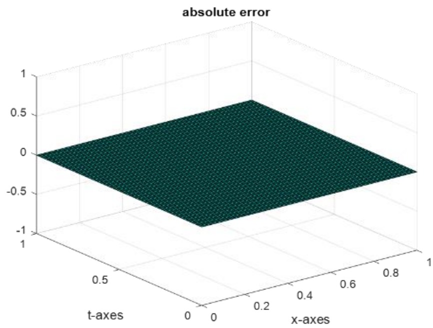


Figure (13) : $0 \leq x \leq 1, 0 \leq t \leq 1$



Figure(14) : $0 \leq x \leq 1, 0 \leq t \leq 1$

6-Conclusion:

It has already been proved that Variation iteration method is a very powerful advice for solving partial differential equations. We had used this method for solving non-linear diffusion problems of different models. The efficiency of this method for solving these problems has been proved. This technique gives an accurate approximation of the exact solution where the obtained accuracy using this method in the studied examples is around five to nine digitals. Using VIM for solving linear and non-linear equations is still a subject of research.

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طريقة التكرارات المتغايرة (VIM) لحل فئة من مسائل الانتشار غير الخطية

خلود أحمد أبوفلغة[1*] ،د.فاطمة التهامي زقوط[2]
[1]قسم الرياضيات، كلية العلوم، جامعة مصراتة، مصراتة، ليبيا
[2]قسم الرياضيات، كلية العلوم، جامعة مصراتة، مصراتة، ليبيا
kholoud.abufalgha@gmail.com

الملخص:
في هذا العمل تم تطبيق طريقة التكرارات المتغايرة ل He . لدراسة ثلاثة نماذج فيزيائية من المعادلات التفاضلية الجزئية غير الخطية، هذه النماذج تصف معادلات الانتشار غير الخطية . باستخدام هذه التقنية تم الحصول على حلول تقريبية وفعلية للمسائل التي تمت دراستها . بعض الأمثلة العددية تم عرضها لنبين فعالية الطريقة ، النتائج المتحصل عليها أظهرت أن طريقة التكرارات المتغايرة أداة قوية وفعالة وبسيطة لحل هذا النوع من المسائل، حيث الدقة المتحصل عليها باستخدام هذه الطريقة تراوحت مابين خمسة إلى تسعة أرقام معنوية للأمثلة المدروسة.

الكلمات المفتاحية: طريقة التكرارات المتغايرة – معادلات الانتشار غير الخطي – المعادلات التفاضلية الجزئية.